

SOLUTION EXAMEN BLANC - ANALYSE 3

Nous vous rappelons que chaque exercice vaut 10 points. Le total des points est donc de 70, et nous comptons uniquement les 7 meilleurs exercices.

Exercise 1.

(i). The function f is clearly well defined and holomorphic on $\mathbb{C} \setminus \{0, 1\}$ and has two simple poles in 0 and 1. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a closed piecewise C^1 path. Then either γ is contractible in $\mathbb{C} \setminus [0, 1]$ or $[0, 1]$ is in the interior of γ . In the first case we have that $\int_{\gamma} f = 0$ by Cauchy's theorem. In the later case we have by the residue theorem that

$$\int_{\gamma} = 2\pi i (\text{ind}(\gamma, 0) \text{Res}(f, 0) + \text{ind}(\gamma, 1) \text{Res}(f, 1)).$$

As clearly $\text{ind}(\gamma, 0) = \text{ind}(\gamma, 1)$ and $\text{Res}(f, 0) = -1$, $\text{Res}(f, 1) = 1$ we have that $\int_{\gamma} f = 0$.

(ii). The function g is clearly well defined, holomorphic and non zero on $U = \mathbb{C} \setminus [0, 1]$. It's logarithmic derivative is given by

$$\frac{g'(z)}{g(z)} = \frac{1}{z} - \frac{1}{z-1} = \frac{-1}{z(z-1)} = -f(z).$$

By (i) we know that $\int_{\gamma} \frac{g'}{g} = 0$ for every closed piecewise C^1 path γ . Thus the map $h(z) := w + \int_{z_*}^z \frac{g'}{g}$, where $w \in \mathbb{C}$ is such that $\exp(w) = g(z_*)$ is well defined and holomorphic as the Morera condition is satisfied. Further as

$$(g \cdot \exp(-h))' = \exp(-h) (g' + g(-h')) = 0.$$

By connectedness of U we get that $g = C \exp(h)$ for some $C \in \mathbb{C}$ and by the choice of $w \in \mathbb{C}$ we get that $g = \exp(h)$.

Exercise 2. Such a function does not exist. Indeed, suppose for contradiction that such an f exists. Then we have that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and by Cauchy's inequality we know $|a_n| \leq \frac{\sup_{\partial D(0,r)} |f|}{r^n}$ for any $r > 0$. But as $\lim_{|z| \rightarrow \infty} \frac{|f(z)|}{|z|^{\frac{3}{2}}}$ we get that

$$\frac{\sup_{\partial D(0,r)} |f|}{r^{\frac{3}{2}}} \xrightarrow{r \rightarrow \infty} 1.$$

We get that

$$|a_n| \leq \frac{r^{\frac{3}{2}} \sup_{\partial D(0,r)} |f|}{r^n}$$

for every $n \in \mathbb{N}$ and $r > 0$. Thus we have that $a_n = 0$ for all $n \geq 2$ and $f(z) = a_0 + a_1 z$. Bu then we clearly have that

$$\lim_{|z| \rightarrow \infty} \frac{|f(z)|}{|z|^{\frac{3}{2}}} = 0,$$

which is a contradiction.

Exercise 3.

(i). Proof of Casorati-Weierstrass.

Theorem 0.1 (Casorati-Weierstrass). *Let $r > 0$ and $f : D(z_*)^* \rightarrow \mathbb{C}$ be holomorphic. If $z_* \in \mathbb{C}$ is an essential singularity, then $f(\mathbb{C}) \subseteq \mathbb{C}$ is dense in \mathbb{C} .*

Proof. Suppose for contradiction that $f(\mathbb{C})$ is not dense. Then there exists $w \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w| \geq \delta$ for all $z \in D(z_*, \delta)^*$. Thus the map $g(z) := \frac{1}{f(z) - w}$ is well defined, holomorphic and bounded by $\frac{1}{\delta}$ and therefore has a removable singularity in z_* . Thus there exists $h : D(z_*, \delta)^* \rightarrow \mathbb{C}$ and $k \in \mathbb{N}$ such that $g(z) = (z - z_*)^k h(z)$ and $h(z) \neq 0$ for all $z \in D(z_*, \delta)$. Bu then we have that $f(z) = \frac{1}{(z - z_*)^k h(z)} + w$ on $D(z_*, \delta)^*$, giving that f has removable singularity or a pole of order k in z_* , which is a contradiction. \square

(ii). To prove this we apply iteratively the open mapping theorem and Casorati-Weierstrass.

Let $\delta_1 > 0$ such that $D(p, \delta_1) \subseteq U$ and let $0 < \delta_2 < \delta_1$. Then $A(p, \delta_2, \delta_1) = D(p, \delta_1) \setminus \overline{D(p, \delta_2)}$ is open and thus its image by f is also open by the open mapping theorem. By Casorati-Weierstrass $f(D(p, \delta_2)^*)$ is dense in \mathbb{C} and therefore $f(D(p, \delta_2)^*) \cap f(A(p, \delta_2, \delta_1)) \neq \emptyset$. Let $z \in D(p, \delta_2)^*$ be such that

$$f(z) \in f(D(p, \delta_2)^*) \cap f(A(p, \delta_2, \delta_1))$$

and define $0 < \delta_3 < |z - p|$. Then clearly

$$f(A(p, \delta_3, \delta_2)) \cap f(A(p, \delta_2, \delta_1)) \neq \emptyset.$$

Further, as $f(D(p, \delta_3)^*)$ is dense in \mathbb{C} there exists $z \in D(p, \delta_3)^*$ such that

$$f(z) \in f(A(p, \delta_3, \delta_2)) \cap f(A(p, \delta_2, \delta_1)).$$

Iteratively, we get for $n \geq 1$ $0 < \delta_{n+1} < \delta_n < \dots < \delta_1$ and $A_k := A(p, \delta_{k+1}, \delta_k)$ with $k = 1, \dots, n$ such that $\bigcap_{k=1}^n f(A_k) \neq \emptyset$. Thus for every $n \in \mathbb{N}$ there exists $z_* \in D(p, \delta_1)^*$ such that $f(z_*)$ is reached at least n -times by f on $D(p, \delta_1)^*$.

Exercise 4. Ceci était exercice 4 de la série 12

Let p, q be two polynomials and assume that q is not the zero map. If q is a constant the claim is trivially respected. Suppose therefore that $\deg(q) \geq 1$. Further, let $\alpha_1, \dots, \alpha_l \in \mathbb{C}$ be the roots of q , i.e. all complex values such that $q(z) = 0$. Then, as q has only finitely many zeros, we have that around each $\alpha \in \{\alpha_1, \dots, \alpha_l\}$ there exists $\delta_\alpha > 0$ such that $q \neq 0$ on $D_{\delta_\alpha}(\alpha)^* := D_{\delta_\alpha}(\alpha) \setminus \{\alpha\}$. Further, as q is a

polynomial, we can write $q(z) = \sum_{n=r_\alpha}^{\deg(q)} a_n(z-\alpha)^n$ with $a_{r_\alpha} \neq 0$ for some $r_\alpha \in \mathbb{N}_{\geq 1}$ as $q(\alpha) = 0$. Therefore, we can write $\frac{p(z)}{q(z)} = \frac{1}{(z-\alpha)^{r_\alpha}} \cdot h_\alpha(z)$ with a holomorphic map on $D_{\delta_\alpha}(\alpha)$. As h_α is holomorphic on $D_{\delta_\alpha}(\alpha)$ there exists a sequence (b_n) of complex values such that $h_\alpha(z) = \sum_{n=0}^{\infty} b_n(z-\alpha)^n$ on $D_{\delta_\alpha}(\alpha)$. We conclude that $\frac{p(z)}{q(z)} - \sum_{n=0}^{r_\alpha-1} b_n(z-\alpha)^{n-r_\alpha} = \sum_{n=0}^{\infty} b_{n+r_\alpha}(z-\alpha)^n$ has a holomorphic extension on $D_{\delta_\alpha}(\alpha)$.

More generally, doing the above around each α_j we get $r_1, \dots, r_l \in \mathbb{N}$ and β_{kj} with $j = 1, \dots, l$ and $k(j) = 1, \dots, r_j$ such that

$$\frac{p(z)}{q(z)} - \sum_{j=1}^l \sum_{k=1}^{r_j} \frac{\beta_{kj}}{(z-\alpha_j)^k}$$

has a holomorphic extension on \mathbb{C} (why?). Let's call this holomorphic extension h and observe that h equals the above expression on $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_l\}$. In particular, h equals the above expression if $|z| \geq M$ for $M > 0$ large enough. But as p is a polynomial there exists clearly $m \in \mathbb{N}$ such that $\frac{h(z)}{z^m} \xrightarrow{|z| \rightarrow \infty} 0$. Thus $|h(z)| \leq C|z|^m$ for some $C > 0$ and all $|z| \geq M$ (why?). Then by the same argument as in exercise 2 sheet 11 we conclude that h is a polynomial of degree at most m . We infer that

$$\frac{p(z)}{q(z)} = h(z) + \sum_{j=1}^l \sum_{k=1}^{r_j} \frac{\beta_{kj}}{(z-\alpha_j)^k}$$

on $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_l\}$ with h a polynomial. The uniqueness is evident as the sum must be exactly the sum of all singular parts of $\frac{p}{q}$. This proves the claim.

Remark 0.2. Observe that if $\deg(p) = \deg(q)$ then h is a non zero constant and if $\deg(q) > \deg(p)$ we have that $h = 0$.

Exercise 5.

(i). The function f is clearly well defined and holomorphic on $\mathbb{C} \setminus M$, where M is the set of the zeros of $1+z^3$. The third roots of unity are given by $1, \exp(\frac{2\pi i}{3}), \exp(\frac{4\pi i}{3})$. Thus the roots of $1+z^3$ are given by $-1, \exp(\frac{\pi i}{3}), \exp(\frac{5\pi i}{3})$. Thus f has three simple poles at $-1, \exp(\frac{\pi i}{3}), \exp(\frac{5\pi i}{3})$.

(ii). For $R > 1$, there is only one pole of f in the interior of γ , which is the one in $\exp(\frac{\pi i}{3})$. As γ is a simple closed path, we have by the residue theorem that

$$\int_{\gamma} f(z) dz = 2\pi i \operatorname{Res}(f, \exp(\frac{\pi i}{3})).$$

Further, by splitting the integral in three parts we get that

$$\int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{\gamma_3} f(z) dz.$$

A parametrization of γ_3 is given by $\gamma_3(t) = (1-t)R \exp(\frac{\pi i}{3})$. Observe that $\gamma_3^3(t) = (1-t)^3 R^3$ and $\gamma_3(t) \cdot \gamma_3'(t) = (1-t)R^2 \exp(\frac{\pi i}{3})$.

$$\begin{aligned} \int_{\gamma_3} f(z) dz &= \exp(\frac{\pi i}{3}) \int_0^1 \frac{(1-t)R^2}{1+(1-t)^3 R^3} dt \\ &= \exp(\frac{\pi i}{3}) \int_0^1 \frac{sR^2}{1+s^3 R^3} dt = \exp(\frac{\pi i}{3}) \int_{\gamma_1} f(z) dz. \end{aligned}$$

We conclude that

$$\begin{aligned} 2\pi i \operatorname{Res}(f, \exp(\frac{\pi i}{3})) &= (1 + \exp(\frac{\pi i}{3})) \int_{\gamma_1} f + \int_{\gamma_2} f \iff \\ \frac{1}{1 + \exp(\frac{\pi i}{3})} \left(2\pi i \operatorname{Res}(f, \exp(\frac{\pi i}{3})) - \int_{\gamma_2} f \right) &= \int_{\gamma_1} f. \end{aligned}$$

(iii). Clearly this integral is well defined and absolutely convergent as $x \mapsto \frac{x}{1+x^3}$ is continuous on $(1, \infty)$ and $\frac{x}{1+x^3} \leq \frac{1}{x^2}$, which is integrable on $[1, \infty)$. As

$$\int_{\gamma_1} = \int_0^R \frac{x}{1+x^3} dx \xrightarrow{R \rightarrow \infty} \int_0^\infty \frac{x}{1+x^3} dx,$$

it is enough to show that $|\int_{\gamma_2}| \xrightarrow{R \rightarrow \infty} 0$ to conclude that

$$\frac{1}{1 + \exp(\frac{\pi i}{3})} \left(2\pi i \operatorname{Res}(f, \exp(\frac{\pi i}{3})) \right) = \int_0^\infty \frac{x}{1+x^3} dx.$$

But this is clearly the case as

$$\left| \int_{\gamma_2} f \right| \leq \frac{2\pi}{3} R \sup_{z \in \gamma_2} \frac{|z|}{|1+z^3|} \leq \frac{2\pi}{3} R \frac{R}{R^3-1} \xrightarrow{R \rightarrow \infty} 0.$$

Finally, as f has a simple pole in $\exp(\frac{\pi i}{3})$ we have that

$$\operatorname{Res}(f, \exp(\frac{\pi i}{3})) = \frac{\exp(\frac{\pi i}{3})}{(\exp(\frac{\pi i}{3}) + 1)(\exp(\frac{\pi i}{3}) - \exp(\frac{5\pi i}{3}))} = \frac{1}{(\exp(\frac{\pi i}{3}) + 1)^2}.$$

Thus we get that

$$\int_0^\infty \frac{x}{1+x^3} dx = \frac{2\pi i}{(\exp(\frac{\pi i}{3}) + 1)^3} = \frac{2\pi}{3\sqrt{3}}.$$

The last equality can be seen by

$$(\exp(\frac{\pi i}{3}) + 1)^3 = 1 + 3(\exp(\frac{\pi i}{3}) + \exp(\frac{2\pi i}{3})) - 1 = 3\sqrt{3}i.$$

Exercise 6.

(i). Let $z_* \in \mathbb{C}$ such that $|z| > 1$, $\delta > 0$ and $D(z_*, \delta) \subset \overline{D(0, 1)}^c$. Let $c > 0$ such that $1 < c < |z|$ for all $z \in D(z_*, \delta)$. Then we have that

$$\left| \frac{z^{2^n}}{z^{2^{n+1}} - 1} \right| \leq \frac{|z|^{2^n}}{(|z|^{2^n} - 1)} \leq \frac{c}{c-1} \frac{1}{c^{2^n} - 1}$$

as $x \mapsto \frac{x}{x-1}$ is monotone decreasing on $(1, \infty)$. As $c^{2^n} \xrightarrow{n \rightarrow \infty} \infty$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $c^{2^n} \geq 2$, giving that

$$\left| \frac{z^{2^n}}{z^{2^{n+1}} - 1} \right| \leq \frac{c}{c-1} \frac{2}{c^{2^n}}$$

for all $n \geq N$ and as clearly $\sum_{n=0}^{\infty} \frac{1}{c^{2^n}} < \infty$ the sum converges locally normally on $\overline{D(0,1)^c}$. Now, let $\overline{D(z_*, \delta)} \subset D(0,1)$ and $c > 0$ be such that $|z| < c < 1$ for all $z \in \overline{D(z_*, \delta)}$. In this case we have that

$$\left| \frac{z^{2^n}}{z^{2^{n+1}} - 1} \right| \leq \frac{c^{2^n}}{1 - |z|^{2^{n+1}}} \leq \frac{c^{2^n}}{1 - c}$$

and as $\sum_{n=0}^{\infty} c^{2^n} < \infty$ the sum converges also locally normally on $D(0,1)$. Thus f is well defined and holomorphic on $\mathbb{C} \setminus \partial D(0,1)$.

(ii). First, we observe that for $z \in \mathbb{C} \setminus \partial D(0,1)$ we have that

$$\begin{aligned} \frac{z^{2^n}}{z^{2^{n+1}} - 1} &= \frac{1}{2} \left(\frac{1}{z^{2^n} - 1} + \frac{1}{z^{2^n} + 1} \right), \\ \frac{1}{z^{2^n} - 1} &= \frac{1}{2} \left(\frac{1}{z^{2^{n-1}} - 1} + \frac{1}{z^{2^{n-1}} + 1} \right). \end{aligned}$$

For $|z| > 1$ the series given by $\frac{1}{z^{2^n \pm 1}}$ converge absolutely. Thus we have that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{1}{z^{2^n} - 1} + \frac{1}{z^{2^n} + 1} \right) &= \sum_{n=0}^{\infty} \frac{1}{z^{2^n} + 1} + \frac{1}{z - 1} + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{z^{2^{n-1}} - 1} + \frac{1}{z^{2^{n-1}} + 1} \right) \\ &= \frac{1}{z - 1} + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{z^{2^n} - 1} + \frac{1}{z^{2^n} + 1} \right). \end{aligned}$$

Thus we get that

$$\sum_{n=0}^{\infty} \frac{z^{2^n}}{z^{2^{n+1}} - 1} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{z^{2^n} - 1} + \frac{1}{z^{2^n} + 1} \right) = \frac{1}{z - 1}.$$

(iii). Let $z \in D(0,1)$. Then there exists a unique $w \in \overline{D(0,1)^c}$ such that $z = \frac{1}{w}$ and we have that

$$f(z) = f\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty} \frac{w^{-2^n}}{w^{-2^{n+1}} - 1} = \sum_{n=0}^{\infty} \frac{w^{2^n}}{1 - w^{2^{n+1}}} = -f(w) = -f\left(\frac{1}{z}\right).$$

Therefore we have that

$$f(z) = -\frac{1}{\frac{1}{z} - 1} = \frac{z}{z - 1}.$$

Exercise 7.

(i). We clearly have that $\sum_{j=0}^{\infty} z^{2^j}$ converge locally normally on $D(0,1)$. As there exists $\delta > 0$ such that $\frac{1}{2}|z| \leq |\operatorname{Ln}(1+z)| \leq \frac{3}{2}|z|$ for all $z \in D(0,\delta)$, we infer that

$\sum_{j=0}^{\infty} \text{Ln}(1 - z^{2^j})$ converge locally normally on $D(0, 1)$ and thus by continuity, we have that

$$\prod_{j=0}^N (1 - z^{2^j}) = \exp \left(\sum_{j=0}^N \text{Ln}(1 - z^{2^j}) \right),$$

converges locally uniformly on $D(0, 1)$. By local uniform convergence of a sequence of holomorphic maps, we infer that $f(z) := \prod_{j=0}^{\infty} (1 - z^{2^j})$ is well defined on $D(0, 1)$ and holomorphic.

(ii). To find the power series of f centered at the origin, we analyse the finite products $\prod_{j=0}^N (1 - z^{2^j})$. The important observation is that this finite product already determines the first $2^{N+1} - 1$ powers of f . We begin by observing that $\prod_{j=0}^N (1 - z^{2^j})$ is given by

$$\begin{cases} 1 - z & \text{if } N = 1, \\ (1 - z)(1 - z^2) = 1 - z - z^2 + z^3 & \text{if } N = 2, \\ (1 - z)(1 - z^2)(1 - z^4) = 1 - z - z^2 + z^3 - z^4 + z^5 + z^6 - z^7 & \text{if } N = 3. \end{cases}$$

From that we infer that

$$\prod_{j=0}^N (1 - z^{2^j}) = \sum_{n=0}^{2^{N+1}-1} (-1)^{s(n)} z^n,$$

where $s(n) = \sum_{k=0}^{\infty} b_k$ and the sequence $(b_k)_{k \in \mathbb{N}} \subseteq \{0, 1\}^{\mathbb{N}}$ is such that $n = \sum_{k=0}^{\infty} b_k 2^k$. It is not difficult to show by induction that this is indeed the case. We already verified the cases $N = 0, 1, 2$. Suppose it holds for $N \geq 0$. Then we have that

$$\begin{aligned} \prod_{j=0}^{N+1} (1 - z^{2^j}) &= (1 - z^{2^{N+1}}) \sum_{n=0}^{2^{N+1}-1} (-1)^{s(n)} z^n = \sum_{n=0}^{2^{N+1}-1} (-1)^{s(n)} z^n + \sum_{n=0}^{2^{N+1}-1} (-1)^{s(n)+1} z^{n+2^{N+1}} \\ &= \sum_{n=0}^{2^{N+1}-1} (-1)^{s(n)} z^n + \sum_{n=2^{N+1}}^{2^{N+2}-1} (-1)^{s(n)} z^n = \sum_{n=0}^{2^{N+2}-1} (-1)^{s(n)} z^n, \end{aligned}$$

where we used that $s(n + 2^{N+1}) = s(n) + 1$ for all $n \in \{0, \dots, 2^{N+1} - 1\}$.

Lastly, we conclude that

$$\sum_{n=0}^{2^{N+1}-1} (-1)^{s(n)} z^n \xrightarrow{N \rightarrow \infty} f(z)$$

for every $z \in D(0, 1)$. Further, as $\sum_{n=0}^{\infty} (-1)^{s(n)} z^n$ is clearly an entire series centered at the origin and with convergence radius $r = 1$, we get that

$$f(z) = \sum_{n=0}^{\infty} (-1)^{s(n)} z^n$$

and by uniqueness of the power series representation of holomorphic functions, we have that this is the Taylor series (or power series) representation of f centered at the origin.

Exercise 8. The poles of the map $f(z) = \frac{1}{(1+z^2)^2(1+2z^2)}$ are given by $\pm i, \pm \frac{i}{\sqrt{2}}$. Further, the map f is clearly even on \mathbb{R} , non negative and the integral is well defined, i.e. converges absolutely. As the map is even we have that $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$. Consider now the closed simple path given by $\gamma = \gamma_1 \cup \gamma_2$, where $\gamma_1(t) = 2Rt - R$ and $\gamma_2(t) = R \exp(\pi ti)$ for $R > 1$. Then by the residue theorem, we have that

$$\int_{\gamma} f = 2\pi i \left(\text{Res}(f, i) + \text{Res}\left(f, \frac{i}{\sqrt{2}}\right) \right) = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

Further, we observe that

$$\left| \int_{\gamma_2} f \right| \leq \pi R \sup_{\gamma_2} |f| \leq \frac{\pi R}{2(R^2 - 1)^2(R^2 - \frac{1}{2})} \xrightarrow{R \rightarrow \infty} 0.$$

As we also clearly have that $\int_{\gamma_1} f \xrightarrow{R \rightarrow \infty} \int_{\mathbb{R}} f$, we get that

$$\int_0^\infty f(x) dx = \frac{1}{2} \int_{\mathbb{R}} f(x) dx = \pi i \left(\text{Res}(f, i) + \text{Res}\left(f, \frac{i}{\sqrt{2}}\right) \right).$$

Now, as f has a simple pole in $\frac{i}{\sqrt{2}}$ we have that

$$\text{Res}\left(f, \frac{i}{\sqrt{2}}\right) = \frac{1}{2(1 - \frac{1}{2})^2 \frac{2i}{\sqrt{2}}} = \frac{1}{\frac{1}{2} \frac{2i}{\sqrt{2}}} = \frac{\sqrt{2}}{i} = -i\sqrt{2}.$$

In i is double, we have

$$\begin{aligned} \text{Res}(f, i) &= \frac{d}{dz} \left[\frac{1}{(z+i)^2(2z^2+1)} \right]_{z=i} = \left[\frac{-2}{(z+i)^3(2z^2+1)} - \frac{4z}{(z+i)^2(2z^2+1)^2} \right]_{z=i} \\ &= \frac{2}{(2i)^3} + \frac{-4i}{-4(-1)^2} = \frac{1}{-4i} + i = \frac{5}{4}i. \end{aligned}$$

We get that

$$\pi i \left(\text{Res}(f, i) + \text{Res}\left(f, \frac{i}{\sqrt{2}}\right) \right) = \pi i \left(\frac{5}{4}i - i\sqrt{2} \right) = \pi \left(\sqrt{2} - \frac{5}{4} \right).$$

We conclude that $\int_0^\infty \frac{1}{(1+x^2)^2(1+2x^2)} dx = \pi \left(\sqrt{2} - \frac{5}{4} \right)$.

Exercise 9. Consider the function given by $h = \exp(f)$. This function is holomorphic on U and such that $|h| = 1$ on $S^1 := \partial D(0, 1)$. By the maximum principal applied to h on $\overline{D(0, 1)}$, we get that $|h| \leq 1$ on $\overline{D(0, 1)}$. Further, as h does not vanish we can apply the minimum principal to infer that $|h| = 1$ on $\overline{D(0, 1)}$. Then, by the open mapping theorem or the again by the maximum principal, we get that h is constant. By continuity of f we infer that f is constant and more precisely that $f = \alpha i$ for some $\alpha \in \mathbb{R}$. Therefore, the maps which are holomorphic on some domain containing $\overline{D(0, 1)}$ and taking only imaginary values on $\partial D(0, 1)$ are exactly the constant maps of the form $f = \alpha i$ for some $\alpha \in \mathbb{R}$.

Exercise 10. *Ceci était exercice 1 de la série 12*

We begin by recalling the definition of local uniform convergence.

Definition 0.3 (Local uniform convergence). Let $V \subseteq \mathbb{C}$ be an open set, $g_n : V \rightarrow \mathbb{C}$ be a sequence of functions and $g : V \rightarrow \mathbb{C}$ be a function. Then we say that g_n converges to g locally uniformly if for every $z \in V$ there exists $\epsilon > 0$ such that $D_\epsilon(z) \subseteq V$ and g_n converges uniformly on $D_\epsilon(z)$ to g .

Remark 0.4. Local uniform convergence as defined above is equivalent to uniform convergence of g_n to g on every compact subset $K \subseteq V$ (why?).

Now, observe that as $f_n : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ is a sequence of holomorphic functions, they are in particular continuous and as they converge locally uniformly to $f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ it is clear that f is continuous (why?). To show that f is even holomorphic, we use the Morera Theorem, which tells us that a continuous function $g : V \subseteq \mathbb{C} \rightarrow \mathbb{C}$ on an open set $V \subseteq \mathbb{C}$ is holomorphic if for every contractible closed path $\gamma : [0, 1] \rightarrow V$ we have that $\int_\gamma g = 0$. But this is clearly satisfied for f as for any contractible path $\gamma : [0, 1] \rightarrow U$ and $n \in \mathbb{N}$ we have $\int_\gamma f_n = 0$ as the f_n are holomorphic and we have

$$\int_\gamma f_n \xrightarrow{n \rightarrow \infty} \int_\gamma f$$

as $\gamma([0, 1]) \subseteq U$ is a compact set and f_n converges uniformly to f on any compact subset of U (why does that imply the convergence of the integral?). We conclude that $f : U \rightarrow \mathbb{C}$ is holomorphic.

Lastly, we want to show that for all $k \in \mathbb{N}$, the sequence of the k -th derivatives converges locally uniformly to $f^{(k)}$. For that we fix $k \in \mathbb{N}$ and $z_* \in U$ and consider $r > 0$ such that $\overline{D_{2r}(z_*)} \subseteq U$. Then we know by Cauchy's integral formula that for any $z \in D_r(z_*)$ and any $n \in \mathbb{N}$ we have that

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D_{2r}(z_*)} \frac{f_n(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{and} \quad f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D_{2r}(z_*)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta.$$

Therefore we have that

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq 2r \cdot k! \sup_{\zeta \in \partial D_{2r}(z_*)} \left| \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^{k+1}} \right| \leq \frac{k!}{r^k} \sup_{\zeta \in \partial D_{2r}(z_*)} |f_n(\zeta) - f(\zeta)|,$$

which tells us that

$$\sup_{z \in D_r(z_*)} |f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k!}{r^k} \sup_{\zeta \in \partial D_{2r}(z_*)} |f_n(\zeta) - f(\zeta)|$$

and shows that $f_n^{(k)}$ converges locally uniformly to $f^{(k)}$ as f_n converges locally uniformly to f .

Remark 0.5. Observe that we only need $U \subseteq \mathbb{C}$ open. We do not need the information that U is connected.

Exercise 11. The entire series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges absolutely for every $z \in \overline{D(0,1)}$. This is trivial as $\frac{|z|^n}{n^2} \leq \frac{1}{n^2}$. Thus the function

$$f : D(0,1) \rightarrow \mathbb{C}, \quad z \mapsto \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is well defined and holomorphic. Furthermore, we have that $f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$. As we know that $\sum_{n=1}^{\infty} \frac{z^n}{n} = -\text{Ln}(1-z)$ on $D(0,1)$, where Ln is the principal branch of the complex logarithm, we get that

$$f'(z) = \frac{1}{z} \sum_{n=1}^{\infty} \frac{z^n}{n} = \frac{-\text{Ln}(1-z)}{z}$$

for every $z \in D(0,1)^*$. As the map

$$g : \mathbb{C} \setminus \{z \in \mathbb{R} | z \geq 1\} \rightarrow \mathbb{C}, \quad \begin{cases} \frac{-\text{Ln}(1-z)}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

is clearly well defined and holomorphic, we have that it is an extension of f' . Finally, as the set $\mathbb{C} \setminus \{z \in \mathbb{R} | z \geq 1\}$ is simply connected, we have that g has a holomorphic primitive G , which can be taken such that $G(0) = 0$. Thus we get that $f = G$ on $D(0,1)$, giving that f has a holomorphic extension on $\mathbb{C} \setminus \{z \in \mathbb{R} | z \geq 1\}$.

Exercise 12. First, observe that

$$\frac{|\alpha z| + |\beta|}{|z|} = |\alpha| + \frac{|\beta|}{|z|} \xrightarrow{|z| \rightarrow \infty} |\alpha|$$

and similarly for $\gamma \in \mathbb{C}$. Thus, there exists $M > 0$ such that for all $|z| \geq M$ we have that $|\alpha z| + |\beta| < |z|$ and similarly for $\gamma \in \mathbb{C}$. Therefore, we have

$$\alpha z + \beta, \alpha z + \gamma \in D(0, M) \quad \forall z \in \partial D(0, M).$$

By the maximum principal we have that $\max_{\overline{D(0,M)}} |f| = \max_{\partial D(0,M)} |f|$. Let $z_* \in \partial D(0, M)$ by a maximizer. Then we have that

$$2|f(z_*)| \leq |f(\alpha z_* + \beta)| + |f(\alpha z_* + \gamma)| \leq 2|f(z_*)|.$$

Thus we have $|f(z_*)| = |f(\alpha z_* + \beta)| = |f(\alpha z_* + \gamma)|$. But then f reaches a local maximum in $\alpha z_* + \beta$ and is therefore constant by the maximum principal.

If the coefficient is changed to something smaller than 2, f is not necessary constant. For example taking $f(z) = z$, $\beta = \gamma = 0$ and $\alpha = \frac{15}{16}$, we get that $\frac{15}{8}f(z) = \frac{15}{8}z$ and $f(\alpha z_* + \beta) + f(\alpha z_* + \gamma) = 2\alpha z_* = \frac{15}{8}z_*$.

Observe also that if we take the coefficient bigger than 2 then $f = 0$.